Design of an enhanced nonlinear PID controller

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Abstract

An enhanced nonlinear PID (EN-PID) controller that exhibits the improved performance than the conventional linear fixed-gain PID controller is proposed in this paper, by incorporating a sector-bounded nonlinear gain in cascade with a conventional PID control architecture. To achieve the high robustness against noise, two nonlinear tracking differentiators are used to select high-quality differential signal in the presence of measurement noise. The criterion to determine the nonlinear gain to retain the stability of the proposed EN-PID control system is addressed, by using the Popov stability criterion. The main advantages of the proposed EN-PID controller lie in its high robustness against noise and easy of implementation. Simulation results performed on a robot manipulator are presented to demonstrate the better performance of the developed EN-PID controller than the conventional fixed-gain PID controller.

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1. Introduction

Proportional-integral-derivative (PID) controllers have been the most popular and the most commonly used industrial controllers in the past years. The popularity and widespread use of PID controllers are attributed primarily to their simplicity and performance characteristics. PID controllers have been utilized for control of diverse dynamical systems ranged from industrial process to aircraft and ship dynamics [1–4].

Although linear fixed-gain PID controllers are often adequate for controlling a nominal physical process, the requirements for high-performance control with changes in operating conditions or environmental parameters are often beyond the capabilities of simple PID controllers. In order to enhance the performance of linear PID controllers, many approaches have been developed to improve the adaptability and robustness by adopting the self-tuning method, general predictive control, fuzzy logic and neural networks strategy, and other methods [5–23].

Amongst these approaches, nonlinear PID (N-PID) control is viewed as one of the most effective and simple method for industrial applications [12–23]. Nonlinear PID control may be any control structure of the following form:

\[ u(t) = k_p(\cdot)e(t) + k_i(\cdot) \int e(\tau) d\tau + k_d(\cdot)\dot{e}(t) \]  

(1)

where \( k_p(\cdot) \), \( k_i(\cdot) \) and \( k_d(\cdot) \) are time-varying controller gains, which may depend on system state, input, or other variables, and \( u(t) \) and \( e(t) \) are the system input and error, respectively.

The nonlinear PID (N-PID) control has found two broad classes of applications:

1. nonlinear systems, where N-PID control is used to accommodate the nonlinearity, usually to achieve consistent response across a range of conditions [12–15]; and
2. linear systems, where N-PID control is used to achieve performance not achievable by a linear PID control, such as increased damping, reduced rise time for step or rapid inputs, improved tracking accuracy, and friction compensation [16–23].

For linear systems, two broad categories of N-PID control are found [16]: those with gains modulated according to the magnitude of the state [22,23], and those with gains modulated according to the phase of the state [16–21]. The former category of N-PID will be used in this paper. According to the magnitude of the state, the enhancement of the controller is achieved by adapting its response based on the performance of the closed-loop control system. When the error between the commanded and actual values of the controlled variable is large, the gain amplifies the error substantially to generate a large correction to rapidly drive the system output to its goal. As the error diminishes, the gain is automatically reduced to prevent excessive oscillations and large overshoots in the response. Because of this automatic gain adjustment, the N-PID controllers enjoy the advantage of high initial gain to obtain a fast response, followed by a low gain to prevent an oscillatory behavior [16].
In this study, a performance enhancement to the conventional linear PID controller is proposed by incorporating a sector-bounded nonlinear gain into a linear fixed-gain PID control architecture. To achieve the high robustness against noise, two nonlinear tracking differentiators are used to select the high-quality differential signal in the presence of measurement noise.

The paper is organized as follows. A nonlinear tracking differentiator is developed in Section 2, which is followed by the development of an enhanced nonlinear PID (EN-PID) controller in Section 3. In Section 4, the stability of the proposed EN-PID controller is analyzed. To verify the performance enhancement, simulations are performed on a two-link revolute robot in Section 5. Finally, some remarking conclusions are summarized in Section 6.

2. Nonlinear tracking differentiator (TD) design

In practice, optical encoder is still the most popular accurate position sensor used in the industrial fields because of its simple detection circuit, high resolution, high accuracy, and relative ease of adaptation in digital control systems. Conversely, the velocity measurement, obtained by means of tachometers, is often contaminated by noise. Furthermore, the use of sensors in the control architecture inherently increases the possibility of failure and increases the cost of the control system due to additional hardware. Therefore, it is necessary to numerically reconstruct velocity signal from position measurement with noise [15,24–28].

The so-called nonlinear tracking differentiator (TD) [15,27,28] is referred to as the following system: given a reference signal \( r(t) \), the system provides two signals \( r_1(t) \) and \( r_2(t) \), such that \( r_1(t) = r(t) \) and \( r_2(t) = \dot{r}(t) \).

For better interpretation of TD, the following lemmas are firstly given.

**Lemma 1.** Suppose \( z(t) \) is a continuous function defined in \([0, \infty)\) that satisfies \( \lim_{t \to \infty} z(t) = 0 \). If \( r(t) = z(Rt) \), \( R > 0 \), then for an arbitrary given \( T > 0 \), the following expression holds

\[
\lim_{R \to \infty} \int_0^T |r(t)| \, dt = 0
\]  

**Proof.** Based on the mean value theorem, there is a \( \tau \) between 0 and \( T \) such that

\[
\int_0^T |r(t)| \, dt = T |r(\tau)| = T |z(R\tau)|
\]  

Since \( \lim_{t \to \infty} z(t) = 0 \), it is straightforward that

\[
\lim_{R \to \infty} \int_0^T |r(t)| \, dt = \lim_{R \to \infty} (T |z(R\tau)|) = 0
\]

Thus Lemma 1 is just established. \( \square \)
Lemma 2. If the solutions to the system (4) hold that $z_1(t) \to 0$ and $z_2(t) \to 0$ as $t \to \infty$

$$
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= f(z_1, z_2)
\end{align*}
$$

then for an arbitrarily constant $c$ and $T > 0$, the solution $r_1(t)$ to the system

$$
\begin{align*}
\dot{r}_1 &= r_2 \\
\dot{r}_2 &= R^2 f\left(r_1 - c, \frac{r_2}{R}\right)
\end{align*}
$$

makes the following expression hold

$$
\lim_{R \to \infty} \int_0^T |r_1(t) - c| \, dt = 0
$$

Proof. Firstly, change the variable of integration as follows:

$$
\begin{align*}
\tau &= \frac{t}{R} \\
r_1(\tau) &= z_1(t) + c \\
r_2(\tau) &= Rz_2(t)
\end{align*}
$$

Based on the mean value theorem, there is a $\tau$ between 0 and $T$ such that

$$
\int_0^T |r_1(t) - c| \, dt = T|r_1(\tau) - c| = T|z_1(t)|
$$

Using Lemma 1, it follows that

$$
\lim_{R \to \infty} \int_0^T |r_1(t) - c| \, dt = \lim_{R \to \infty} (T|z_1(t)|) = T \lim_{R \to \infty} |z_1(R\tau)| = 0
$$

Therefore, Lemma 2 is right justified. □

Theorem 1 [27]. If the arbitrary solutions to system (4) satisfy $z_1(t) \to 0$ and $z_2(t) \to 0$ as $t \to \infty$, then for any arbitrarily bounded integrable function $r(t)$ and given constant $T > 0$, the solution $r_1(t)$ to the system

$$
\begin{align*}
\dot{r}_1 &= r_2 \\
\dot{r}_2 &= R^2 f\left(r_1 - r, \frac{r_2}{R}\right)
\end{align*}
$$

satisfies

$$
\lim_{R \to \infty} \int_0^T |r_1(t) - r(t)| \, dt = 0
$$

Proof. The proof can be justified by dividing into the following two cases:

Case 1: If $r(t)$ is a constant function, then Theorem 1 is Lemma 2.
Case 2: If \( r(t) \ (t \in [0, T]) \) is a bounded integral function, then it is an element of \( L^1[0, T] \), where \( L^1[0, T] \) denotes the set of all first-integrable function in the range of \([0, T] \). For an arbitrarily given \( \varepsilon > 0 \), there exists a simple series \( \varphi_n(t) \ (n = 1, 2, \ldots) \) that uniformly converges to a continuous function \( \varphi(t) \in C[0, T] \) such that [27]

\[
\int_0^T |r(t) - \varphi(t)| dt < \frac{\varepsilon}{2}
\]

Thus, there exists an integer \( N_0 \) such that \( |\varphi(t) - \varphi_M(t)| < \frac{\varepsilon}{4T} \) for all \( M > N_0 \). Consequently, the following inequality holds

\[
\int_0^T |r(t) - \varphi_M(t)| dt \leq \int_0^T |r(t) - \varphi(t)| dt + \int_0^T |\varphi(t) - \varphi_M(t)| dt < \frac{\varepsilon}{2}
\]

Since \( \varphi(t) \) is a continuous function, the simple series \( \varphi_M(t) \) that partitions the range \([0, T] \) into some bounded intervals denoted by \( l_i \ (i = 1, 2, \ldots, m) \). Selecting \( \varphi_M(t) \) to be a deterministic constant in each bounded interval, and based on Lemma 2, there exists \( R_0 > 0 \) such that

\[
\int_{l_i} |r_1(t) - \varphi_M(t)| dt < \frac{\varepsilon}{2m}, \quad i = 1, 2, \ldots, m
\]

for all \( R > R_0 \). Then,

\[
\int_0^T |r_1(t) - \varphi_M(t)| dt < \frac{\varepsilon}{2}
\]

Thereby, the following inequality holds

\[
\int_0^T |r_1(t) - r(t)| dt < \int_0^T |r_1(t) - \varphi_M(t)| dt + \int_0^T |\varphi_M(t) - r(t)| dt < \varepsilon
\]

for all \( R > R_0 \).

The proof is right justified. \( \square \)

Theorem 1 shows that \( r_1(t) \) averagely converges to \( r(t) \). If the bounded integrable function \( r(t) \) is viewed as a generalized function, then \( r_2(t) \) weakly converges to the generalized derivative of \( r(t) \). Therefore, system (9) can be used as a nonlinear tracking differentiator to provide a smooth approach to the original generalized function and its generalized derivative in the sense of average convergence and weak convergence, respectively.

One feasible second-order TD for the reference signal can be expressed as [15,27]

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -R \text{sat}\left(z_1 - r + \frac{z_2}{2R}, \delta\right)
\end{align*}
\]

where \( R \) and \( \delta \) are two positive design parameters, and \( \text{sat}(A, \delta) \) is the following non-linear saturation function:
in which \( \text{sgn}(\cdot) \) stands for a standard signum function.

The parameters \( R \) and \( \delta \) can be determined empirically. \( R \) is named as “velocity factor”, and \( \delta \) “filtering factor”, respectively. Large \( R \) is helpful to fast the transition and tracking, and large \( \delta \) is helpful to cancel out noise. In general, for the reference signal corrupted with maximum amplitude of 0.01 white noise component, \( R \) can be chosen from the range of \([2.5 \ 50]\); for the reference signal with large noise magnitude than 0.01, \( R \) will be increased to preserve a better tracking performance at the sacrifice of differential signal obtained. There is a tradeoff between the tracking and filtering.

The better performance of TD is shown in Fig. 1. The reference input is \( r(t) = \sin(t) \) mm and it is perturbed by an additive white noise component with the maximum amplitude of 0.01 mm. For comparison, the differential signal obtained by the conventional backward differentiator is also shown in Fig. 2. The simulations were programmed in Matlab with a fourth-order Runge–Kutta and run on a PC Pentium III-860. The simulation step is determined as \( h = 0.0025 \), and the initial values of \( z_1 \) and \( z_2 \) is \( z_1(0) = 0 \) and \( z_2(0) = 0 \). The parameters of the TD are determined as \( R = 20 \), and \( \delta = 0.001 \). It can be seen that the differential tracking

\[
\text{sat}(A, \delta) = \begin{cases} 
\text{sgn}(A), & |A| > \delta \\
\frac{A}{\delta}, & |A| \leq \delta 
\end{cases}
\] (17)
performance of TD is much better over that of the conventional backward difference method.

3. Enhanced nonlinear PID (EN-PID) controller

The proposed enhanced nonlinear PID (EN-PID) controller consists of a sector-bounded nonlinear gain \( k(e) \), a linear fixed-gain PID controller expressed by \( K(s) = k_p + k_i/s + k_d s \), and two nonlinear tracking differentiators (TDs), as shown in Fig. 3. \( k_p \), \( k_i \), and \( k_d \) are proportional, integral, and derivative gains, which can be determined by the Ziegler–Nichols criterion [1]. The nonlinear gain \( k(e) \) is a sector-bounded function of the error \( e(t) \), and acts on the error to produce the “scaled” error \( f(e) = k(e) \cdot e(t) \). Using the high-quality differential signal selected by the developed TDs, we have the following enhanced nonlinear N-PID control law:

\[
\begin{align*}
u(t) &= \left[ k_p + k_i \int_0^t dt + k_d \frac{de}{dt} \right] \cdot f(e) \\
&= k_p[k(e) \cdot e(t)] + k_i \int_0^t [k(e) \cdot e(t)] dt + k_d[k(e) \cdot c(t)]
\end{align*}
\]

where \( e(t) \) and \( c(t) \) are expressed as, respectively,

\[ e(t) = \text{Reference} - \text{Obtained by backward difference} \]

![Fig. 2. Performance of backward differentiator with noise.](image-url)

Fig. 3. Block diagram of the EN-PID control system.
The nonlinear gain $k(e)$ represents any general nonlinear function of the error $e$ which is bounded in the sector $0 \leq k(e) \leq k_{\text{max}}$. There is a broad range of options available for the nonlinear gain $k(e)$ [21]. One simple form of the nonlinear gain function can be expressed as

$$k(e) = \text{ch}(k_0 e) = \frac{\exp(k_0 e) + \exp(-k_0 e)}{2}$$

$$e(t) = z_1(t) - z_3(t)$$
$$c(t) = z_2(t) - z_4(t)$$

$$e = \begin{cases} e & |e| \leq e_{\text{max}} \\ e_{\text{max}} \text{sgn}(e) & |e| > e_{\text{max}} \end{cases}$$

where $k_0$ and $e_{\text{max}}$ are user-defined positive constants. The nonlinear gain $k(e)$ is lower-bounded by $k(e)_{\text{min}} = 1$ when $e = 0$, and upper-bounded by $k(e)_{\text{max}} = \text{ch}(k_0 e_{\text{max}})$. Therefore, $e_{\text{max}}$ denotes the range of variation, and $k_0$ defines the rate of variation of $k(e)$. Fig. 4 illustrates a typical variation of $k(e)$ with respect to $e$ when $k_0 = 0.125$ and $k_0 = 0.15$, with $e_{\text{max}} = 10$. It can be seen that the gain $k(e)$ is an even function of $e$, that is, $k(-e) = k(e)$.

As a result, the proposed EN-PID controller can be realized by cascading this sector-bounded nonlinear gain $k(e)$ with the available linear fixed-gain PID controller, using the high-quality differential signal selected by the developed TDs.

4. Determination of nonlinear gain

In many applications, the dynamics of the system to be controlled can be adequately modeled by a second-order differential equation. Even when the system dynamics is of high order, the response of the system is often largely dependent
on the location of a pair of dominant complex poles, which can be embodied in a second-order model [21,29]. For these systems, the second-order transfer function relating the system output \( y(t) \) to the control action \( u(t) \) can be expressed as

\[
G(s) = \frac{y(s)}{u(s)} = \frac{\omega_n^2 k}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{c}{s^2 + as + b}
\]  

(22)

where \( \xi, \omega_n, \) and \( k \) are the damping ratio, natural frequency, and DC gain of the controlled system, respectively, \( a = 2\xi\omega_n, b = \omega_n^2, \) and \( c = \omega_n^2k. \)

For such a control system, it has been proven that there exists a suitable choice of the PID controller parameters so that the overall closed-loop system is globally asymptotically stable for the position (set-point) control [30–32]. Since the convergence of the developed nonlinear tracking differentiator (TD) has just been investigated, in this section, the criterion to determine the nonlinear gains to retain the stability of the proposed EN-PID control system that incorporates a sector-bounded nonlinear gain in cascade with a stable linear fixed-gain PID control system is addressed, following the idea presented by Seraji [21] using the Popov stability criterion. The Popov stability criterion can be stated graphically as follows: A sufficient condition for the closed-loop system to be global asymptotically stable for all nonlinear gains in the sector \( 0 \leq k(e) \leq k(e)_{\text{max}} \) is that the Popov plot of \( W(j\omega) \) lies entirely to the right of a straight line with a nonnegative slope passing through the point \( -1/k_{\text{max}} + j0 \) [33,34].

For some applications, it is desirable to introduce the sector-bounded nonlinear gain \( k(e) \) on some terms of the linear fixed-gain PID controller, instead of equally on all three terms. In this section, we will investigate the criterion to determine the nonlinear gains for the three complex cases: the enhanced nonlinear PI, PD, and PID controllers, by using the Popov stability criterion, respectively. The criterion to determine the nonlinear gains for other partially enhanced nonlinear P, I, and D controller can be analyzed in a similar way.

4.1. Determination of nonlinear gains for EN-PI controllers

In this case, the enhanced nonlinear PI control system can be viewed as the following linear third-order transfer function \( W(s) \):

\[
W(s) = K(s)G(s) = \frac{c(k_p s + k_i)}{s(s^2 + as + b)}
\]  

(23)

which describes the controlled plant (22) and the linear fixed-gain PI controller cascades with the sector-bounded nonlinear gain \( k(e). \) \( k_p \) and \( k_i \) are positive constant proportional and integral gains of the linear PI controller, respectively.

To apply the Popov stability criterion, the crossing of the Popov plot of \( W(j\omega) \) with the real axis should be computed. This plot reveals the range of values that the sector-bounded nonlinear gain \( k(e) \) can be assumed while retaining the closed-loop stability. From (23), the real and imaginary parts of \( W(j\omega) \) can be calculated [21,34]
\[
\text{Re} W(j\omega) = \frac{-c(k_p \omega^2 + ak_i - bk_p)}{a^2 \omega^2 + (b - \omega^2)^2} 
\]
\[
\omega \text{ Im} W(j\omega) = \frac{-c[(ak_p - k_i)\omega^2 + bk_i]}{a^2 \omega^2 + (b - \omega^2)^2} 
\]

The Popov plot of \(W(j\omega)\) starts at the point \(P(-c(ak_i - bk_p)/b^2, -ck_i/b)\) for \(\omega = 0\) and terminates at the point \(Q(0,0)\) for \(\omega = \infty\). Two cases are now possible, depending on the relative values of \(k_p\) and \(k_i\).

**Case 1:** \(k_i \leq ak_p\). In this case, \(\omega \text{ Im} W(j\omega)\) is always negative for all \(\omega\), that is, the Popov plot of \(W(j\omega)\) remains entirely in the third and fourth quadrants and does not cross the real axis. This implies that one can construct a straight line with a nonnegative slope passing through the origin such that the Popov plot is entirely to the right of this line. Therefore, according to the Popov stability criterion [33,34], the range of the allowable nonlinear gain \(k(e)\) is \((0,\infty)\). A typical Popov plot for \(a = 3\), \(b = 2\), and \(c = 10\), with \(k_p = 1\) and \(k_i = 0.5\), is shown in Fig. 5a.

**Case 2:** \(k_i > ak_p\). The crossover frequency \(\omega_0\) of \(W(j\omega)\) to the real axis can be found by solving \(\omega \text{ Im} W(j\omega) = 0\) to yield [21,34]

\[
\omega_0^2 = \frac{bk_i}{k_i - ak_p} 
\]

As a result, the value of \(W(j\omega)\) at the crossover is

\[
\text{Re} W(j\omega_0) = \frac{c(ak_p - k_i)}{ab} 
\]

It indicates that the Popov plot crosses the negative real axis. So, the maximum allowable nonlinear gain can be calculated as [21]

\[
k(e)_{\text{max}} = -\frac{1}{\text{Re} W(j\omega_0)} = \frac{ab}{c(ak_p - k_i)} 
\]

Then a straight line with a nonnegative slope passing through the point \(-1/k_{\text{max}} + j0\) can be constructed such that the Popov plot of \(W(j\omega)\) is entirely to the right of this line.

![Fig. 5. Popov plot for the EN-PI control system: (a) Case 1 and (b) Case 2.](image-url)
The range of the allowable nonlinear gain \( k(e) \) is \((0,k(e)_{\text{max}})\). A typical Popov plot for \( a=3, b=2, c=10, \) with \( k_p = 0 \) and \( k_i = 0.5 \), is shown in Fig. 5b.

### 4.2. Determination of nonlinear gains for EN-PD controllers

In this case, the closed-loop system can be separated out the nonlinear gain \( k(e) \) and a second-order transfer function \( W(s) \) which consists of the PD controller and the plant (22), that is,

\[
W(s) = K(s)G(s) = \frac{c(k_p + k_ds)}{s^2 + as + b}
\]  

(29)

From (29), the following expressions of Popov plot \( W(j\omega) \) can be obtained [21,34]

\[
\text{Re} W(j\omega) = \frac{c[(ak_d - k_p)\omega^2 + bk_p]}{a^2\omega^2 + (b - \omega^2)^2}
\]  

(30)

\[
\omega \text{Im} W(j\omega) = \frac{-co\omega^2(k_d\omega^2 + ak_p - bk_d)}{a^2\omega^2 + (b - \omega^2)^2}
\]  

(31)

It can be seen that, the Popov plot of \( W(j\omega) \) starts at the point \( P(ck_p/b,0) \) for \( \omega = 0 \) and terminates at the point \( Q(0,-ck_d) \) for \( \omega = \infty \). Two distinct cases are now possible, depending on the relative values of \( k_p \) and \( k_d \).

**Case 1:** \( bk_d \leq ak_p \). In this case, from (31) it can be seen that \( \omega \text{Im} W(j\omega) \) is always negative for all nonzero \( \omega \), that is, the Popov plot of \( W(j\omega) \) remains entirely in the third and fourth quadrants and does not cross the real axis. Therefore, according to the Popov stability criterion [33,34], the range of the allowable nonlinear gain \( k(e) \) is \((0,\infty)\). Fig. 6a illustrates a typical Popov plot for \( a=3, b=2, \) and \( c=10, \) with \( k_p = 1 \) and \( k_i = 0.5 \).

**Case 2:** \( bk_d > ak_p \). The crossover frequency \( \omega_0 \) of \( W(j\omega) \) to the real axis is found by solving \( \omega \text{Im} W(j\omega) = 0 \) to yield

\[
\omega_0^2 = \frac{bk_d - ak_p}{k_d}
\]  

(32)

![Fig. 6. Popov plot for the EN-PD control system: (a) Case 1 and (b) Case 2.](image)
and the value of $W(j\omega)$ is then found to be

$$\text{Re} \, W(j\omega) = \frac{ck_d}{a}$$

which indicates that the Popov plot crosses the positive real axis. The general shape of the Popov plot can be seen from a typical case shown in Fig. 6b, where $a = 3$, $b = 2$, and $c = 10$, with $k_p = 0$ and $k_d = 0.5$. It can be seen that it is possible to construct a straight line with a positive slope passing through the origin such that the Popov plot is entirely to the right of this line. Hence, the range of the allowable nonlinear gain $k(e)$ is $(0, \infty)$.

It can be summarized that, in these two cases, the stability of the enhanced nonlinear PD controlled system can be always guaranteed with the unbounded nonlinear gain $k(e)$.

### 4.3. Determination of nonlinear gains for EN-PID controllers

In this case, the closed-loop system can be viewed as a third-order transfer function $W(s)$ cascades a nonlinear gain $k(e)$. The third-order transfer function $W(s)$ can be expressed as

$$W(s) = K(s)G(s) = \frac{c(k_ds^2 + k_ps + k_i)}{s(s^2 + as + b)}$$

where $k_p$, $k_i$, and $k_d$ are the positive constant proportional, integral, and derivative gains, respectively.

From (34), the real and imaginary parts of $W(j\omega)$ can be expressed as [21,34]

$$\text{Re} \, W(j\omega) = \frac{c[(ak_d - k_p)\omega^2 + bk_p - ak_i]}{a^2\omega^2 + (b - \omega^2)^2}$$

$$\omega \text{Im} \, W(j\omega) = -\frac{c[k_d\omega^4 + (ak_p - bk_d - k_i)\omega^2 + bk_i]}{a^2\omega^2 + (b - \omega^2)^2}$$

The Popov plot of $W(j\omega)$ starts at the point $P(-c(ak_i - bk_p)/b^2, ck_d/b)$ for $\omega = 0$ and terminates at the point $Q(0,-ck_d)$ for $\omega = \infty$. To apply the Popov stability criterion, the crossing of the Popov plot of $W(j\omega)$ with the real axis must be determined. From (36), it is clear that when $(ak_p - bk_d - k_i) \geq 0$, or

$$bk_d + k_i \leq ak_p$$

then $\omega \text{Im} \, W(j\omega)$ is negative for all $\omega$; thus the Popov plot does not cross the real axis. In this case, the range of the nonlinear gain $k(e)$ for stability is $(0, \infty)$. Hence (37) gives a sufficient, but not a necessary condition for closed-loop stability for all values of $k(e)$.

When $bk_d + k_i > ak_p$, the closed-loop system may become unstable for some values of $k(e)$. These values of $k(e)$ correspond to the cases where the Popov plot crosses the real axis, that is, $\omega \text{Im} \, W(j\omega) = 0$. Therefore, two distinct cases are possible, depending on the relative values of $k_p$, $k_i$ and $k_d$. 
Case 1: $\sqrt{ak_p} \leq |\sqrt{bk_d} - \sqrt{k_i}|$. In this case, the two crossover frequencies of the Popov plot of $W(j\omega)$ to the real axis, $\omega_1^2$ and $\omega_2^2$, where $\omega_1^2 < \omega_2^2$. These frequencies are the roots of the following equation [21]:

$$(k_d\omega^2 - k_i)(\omega^2 - b) + ak_p\omega^2 = 0$$

(38)

The values of $W(j\omega)$ at the two crossovers are then found from (35) as

$$\Re W(j\omega_n) = \frac{c[(ak_d\omega_n^2 - k_i) - k_p(\omega_n^2 - b)]}{a^2\omega_n^2 + (b - \omega_n^2)^2} \quad n = 1, 2$$

(39)

Substituting for $(\omega_n^2 - b)$ from (38) into (39) and simplifying the result yields

$$\Re W(j\omega_n) = \frac{ck_p}{b - \omega_n^2} \quad n = 1, 2$$

(40)

Now for the Popov plot to cross the negative axis, we need to find the condition under which $b < \omega_n^2$. Consider the following polynomial:

$$g(\omega_n^2) = k_d\omega_n^4 + (ak_p - bk_d - k_i)\omega_n^2 + bk_i$$

(41)

where the plot of $g(\omega_n^2)$ versus $\omega_n^2$ is a parabola that cross the $\omega_n^2$-axis at $\omega_1^2$ and $\omega_2^2$. Since $k_d > 0$, for any value of $\omega_n^2$ “inside” the parabola, the expression $g(\omega_n^2)$ is negative, whereas for all values of $\omega_n^2$ “outside” the parabola (including the origin $\omega_n^2 = 0$), the expression $g(\omega_n^2)$ is positive. For $\omega_n^2 = b$, we have $g(b) = abk_p > 0$, hence $\omega_n^2 = b$ is located outside the parabola, that is, either $b < \omega_1^2 < \omega_2^2$ or $\omega_1^2 < b < \omega_2^2$.

To find out the condition needed for the former case to occur, we only need to compare the location of the midpoint $\omega_0^2 = (\omega_1^2 + \omega_2^2)/2$ relative to $b$. For $b < \omega_1^2$, we need $b < \omega_0^2$. Using the sum-of-roots relationship for (38) yields [21]

$$b < -\frac{ak_p - k_i - bk_d}{2k_d}$$

(42)

which can be simplified to

$$ak_p + bk_d < k_i$$

(43)

Therefore, we can conclude that when $\sqrt{ak_p} \leq |\sqrt{bk_d} - \sqrt{k_i}|$ and $ak_p + bk_d < k_i$, the Popov plot of $W(j\omega)$ crosses the negative real axis [$\Re W(j\omega_1) < 0$], and the nonlinear gain $k(e)$ must be upper-bounded by

$$k(e)_{\text{max}} = -\frac{1}{\Re W(j\omega_1)} = \frac{\omega_1^2 - b}{ck_p}$$

(44)

to ensure closed-loop stability, that is, $0 \leq k(e) \leq k(e)_{\text{max}}$. Notice that since $\omega_1^2 < \omega_0^2$, from (44), the maximum nonlinear gain is bounded by

$$k_{\text{max}} < \frac{k_i - ak_p - bk_d}{2ck_pk_d}$$

(45)

For this case, a typical Popov plot when $a = 3$, $b = 2$, and $c = 10$, with $k_p = 0$, $k_i = 1$ and $k_d = 1$, is illustrated in Fig. 7a.
On the other hand, when \( \sqrt{ak_p} \leq |\sqrt{bk_d} - \sqrt{k_i}| \) but \( ak_p + bk_d \geq k_i \), the Popov plot of \( W(j\omega) \) crosses the positive real axis \( \text{Re} \, W(j\omega) > 0 \), and the general shape of the Popov plot is shown in Fig. 7a. It can be seen that it is possible to construct a straight line with a positive slope passing through the origin such that the Popov plot is entirely to the right of this line. Hence from the Popov stability criterion, the nonlinear gain \( k(e) \) is \([0, +\infty)\).

Case 2: \( \sqrt{ak_p} > |\sqrt{bk_d} - \sqrt{k_i}| \). In this case, (36) cannot have positive real roots for \( \omega \). Hence, the Popov plot of \( W(j\omega) \) does not cross the real axis and remains entirely in the third and fourth quadrants. Therefore, according to the Popov stability criterion \([33,34]\), the range of the allowable nonlinear gain \( k(e) \) is \((0, \infty)\). Fig. 7b illustrates a typical Popov plot in this case for \( a = 3, b = 2, \) and \( c = 10, \) with \( k_p = 1, k_i = 1 \) and \( k_d = 1 \).

From the condition \( ak_p + bk_d < k_i \), we conclude that the effect of increasing the derivative gain \( k_d \) is to increase the range of the integral gain \( k_i \) for stability.

5. Simulation results

Numerical simulations on a two-link revolute-joint manipulator shown in Fig. 8, were performed to validate the effectiveness of the developed EN-PID controller.

The dynamics of the two-link revolute-joint manipulator are given by [35]

\[
\begin{align*}
\tau_1 &= m_2l_2^2(q_1 + q_2) + m_2l_1l_2c_2(2q_1 + q_2) + (m_1 + m_2)l_2^2(q_1 - 2m_2l_1l_2s_2q_1q_2) \\
&- m_2l_1l_2s_2^2\\
\tau_2 &= m_2l_1l_2c_2q_1 + m_2l_2^2(q_1 + q_2) + m_2l_1l_2s_2q_1^2 + m_2l_2gc_2 + f_1(q_1) + T_d
\end{align*}
\]

where \( s_2 = \sin(q_2), c_2 = \cos(q_2), c_1 = \cos(q_1), c_{12} = \cos(q_1 + q_2), q_1 \) and \( q_2 \) are link angles, \( m_1 \) and \( m_2 \) are link masses, \( l_1 \) and \( l_2 \) are link lengths, and \( \tau_1 \) and \( \tau_2 \) are the output torques of motors reflected to the joint axes, respectively. \( f_1(q_1), f_2(q_2), \) and \( T_d \) are the viscous friction and disturbed torque, respectively. They can be expressed as
\( f_1(\dot{q}_1) = 0.5 \text{sgn}(\dot{q}_1), \quad f_2(\dot{q}_2) = 0.5 \text{sgn}(\dot{q}_2), \quad T_d = 5 \cos(5t) \) (47)

Assume that the parameters are known and given as follows. \( m_1 = m_2 = 1.0 \text{ kg}, \) and \( l_1 = l_2 = 1.0 \text{ m}. \) We compare the performance of the proposed EN-PD controller and the conventional fixed-gain PD controller. For the two PD controllers, the following identical gains are used: \( k_p = 5500, \) \( k_d = 60. \) The design parameters for the sector-bounded nonlinear gain \( k(e) \) is chosen as \( k_0 = 200 \) and \( e_{\text{max}} = 0.006. \) The
parameters of TDs are determined as the same as that in Section 2. The sampling period is selected as $h = 0.0025$ s. The initial conditions are all chosen to be zero. The desired trajectory is $q_{1d} = q_{2d} = (1 - \cos(t))$ rad.

The link trajectory and its tracking error are shown in Figs. 9 and 10, respectively, using the conventional PD controller and EN-PD controller. It can be seen from comparison of Fig. 9 with Fig. 10, that the tracking error of EN-PD controller is much less than that obtained by the conventional fixed-gain PD controller. The variation of the nonlinear gain is shown in Fig. 11. It is clear that when there are large errors, the nonlinear gain rises from 1 to 1.67 and 1.17 for the two EN-PD controllers, respectively, so that a large corrective action can be generated to rapidly drive the system output to its goal. Therefore, it can be concluded that the favorable result mainly comes from the nonlinear gain variation of the proposed EN-PD control.

To validate the high robustness against noise and the contribution of the developed nonlinear tracking differentiator (TD), an additive noise with the magnitude of 0.001 rad is added to the feedback position signal. For limitation of space, only the position tracking error is presented in Figs. 12–14, respectively, by using the conventional PD controller, the nonlinear PD (N-PD) controller without the developed TD, and the EN-PD controller. From the comparison results, it can be seen that the respond speed of the developed EN-PD controller is much faster than that of the conventional fixed-gain PD controller, and lower tracking error is obtained. It can conclude that the developed EN-PD controller has better robustness against noise.

![Fig. 10. Simulation result of the EN-PD controller.](image-url)
over the conventional fixed-gain PD controller, and the developed TD has laid a solid base for the better performance.
6. Conclusions

To enhance the performance of a conventional fixed-gain PID controlled system, an enhanced nonlinear PID (EN-PID) controller is proposed, by incorporating a sector-bounded nonlinear gain function and two nonlinear tracking differentiators (TDs) into the available fixed-gain PID control architecture. The nonlinear gain is a function of the error, and acts on the error to produce the “scaled” error for fast response. The proposed nonlinear tracking differentiators select a high-quality differential signal in the presence of measurement noise. The EN-PID controller can be implemented by cascading a nonlinear gain with an available fixed-gain PID controller without re-determination of the parameters of the PID controller. The stability of the proposed EN-PID controllers is analyzed using the Popov criterion, and the appreciate selection of the nonlinear gain to guarantee the stability and performance enhancement is addressed. The comparison of the proposed EN-PD controller with a conventional fixed PD controller on a two-link revolute-joint manipulator is performed to verify the effectiveness. The major significance of the proposed EN-PID controller lies in its high robustness against noise and superior performance over the conventional fixed-gain PID controller, and easy of engineering implementation, which explores a convenient engineering method to improve the performance of the available conventional linear fixed-gain PID control system.

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